

Separating maps between Lebesgue-Fourier algebras

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Abstract

Let G_1 and G_2 be locally compact groups; it is known that $\mathcal{LA}(G_i) = L^1(G_i) \cap A(G_i)$ is an abstract Segal algebra with respect to $A(G_i)$ for $i = 1, 2$. A linear mapping $T : \mathcal{LA}(G_1) \rightarrow \mathcal{LA}(G_2)$ is a *separating* map if $f \cdot g \equiv 0$ implies $Tf \cdot Tg \equiv 0$ for $f, g \in \mathcal{LA}(G_1)$. In this paper, we show that a separating bijective map is always continuous and also, that there exist some extensions of T to the larger algebras. We introduce a certain condition (condition (P)) under which the existence of a bijective separating map leads to existence of a topological isomorphism between G_1 and G_2 . We also characterize bijective separating maps as a weighted isomorphism on locally compact amenable groups. Moreover, we derive some similar results for Lebesgue-Fourier algebras considered as Segal algebras for locally compact abelian groups.¹

1 Preliminaries and introduction

Let G be a locally compact group with left Haar measure λ and let $L^1(G)$ be the group algebra of G as defined in [13] endowed with the norm $\|\cdot\|_1$ and the convolution product $*$. Let also $A(G)$ denote the Fourier algebra of G as defined by Eymard [7]. Write $\mathcal{LA}(G) := L^1(G) \cap A(G)$ and define

$$|||h||| := \|h\|_1 + \|h\|_{A(G)} \quad (h \in \mathcal{LA}(G)).$$

Then $\mathcal{LA}(G)$ with norm $|||\cdot|||$ is a Banach space; this space was studied extensively by Ghahramani and Lau in [12]. They show that $\mathcal{LA}(G)$ with the convolution product is a Banach algebra and call it the *Lebesgue-Fourier algebra* of G ; moreover, it is a Segal algebra for locally compact group G . Also, $\mathcal{LA}(G)$ with pointwise multiplication is a Banach algebra and even an abstract Segal algebra with respect to $A(G)$, (see [12]).

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Separating maps (also considered under the name of *disjointness preserving maps*) between general vector lattices were studied by several authors, for example [1, 3]. Separating maps were later considered in [4] for spaces of continuous functions defined on compact Hausdorff spaces. Subsequently, several various results were gained on locally compact groups. For example, Font and Hernandez studied separating maps between the algebra of bounded continuous maps on locally compact groups in [9, 11]. In [10], they used separating maps between Fourier algebras of two locally compact abelian groups to study the separating maps between group algebras of locally compact abelian groups. Later on, the results gained in [10] led to a study on separating maps between Fourier algebras of two arbitrary locally compact groups (not necessarily abelian) in [8]. In [15], Monfared obtained a comprehensive characterization for separating maps on Fourier algebras of locally compact groups.

Recently, Alaminos, Bresar, Extremera, and Villena in [2] attained a characterization for the continuous separating maps between group algebras of locally compact groups. Their study also led to a characterization of continuous separating maps between C^* -algebras. On the other hand, since $C_0(G)$ and $C_b(G)$, respectively the bounded continuous functions that vanish at infinity and bounded continuous functions on locally compact group G , form two C^* -algebras, these results would apply to them.

In this paper we study the separating maps between Lebesgue-Fourier algebras of locally compact groups. After proving two lemmas in this section, in section 2 we construct the main tools for studying such maps. This part will be followed by section 3 in which we prove some properties of the *bijective separating maps*. We derive extensions for the bijective separating maps between Lebesgue-Fourier algebras of locally compact groups G_1 and G_2 to bijective separating maps between $A(G_1)$, $A(G_2)$ and to ones between $C_0(G_1)$, $C_0(G_2)$. In section 4, introducing the condition (P) , we study topological isomorphism between locally compact groups. Section 5 focuses on amenable locally compact groups. In this section we show that for amenable locally compact groups every bijective separating map T from $\mathcal{LA}(G_1)$ onto $\mathcal{LA}(G_2)$ is the composition of an algebra isomorphism of $\mathcal{LA}(G_1)$ onto $\mathcal{LA}(G_2)$ and a linear transformation on $\mathcal{LA}(G_1)$ into itself; indeed, we can say that T is a weighted isomorphism, see [2]. In section 6 we define separating maps on Lebesgue-Fourier algebras with convolution multiplication and pursue our studying while Lebesgue-Fourier algebras are considered as Segal algebras of locally compact abelian groups.

Lemma 1.1 *Let G be a locally compact group and K be a compact subset of G , and*

let U be an open subset of G such that $K \subset U$. For each V a relatively compact open neighborhood of e such that $KVV \subseteq U$ we can find f_V in $\mathcal{LA}(G) \cap C_c(G)$ such that

- (i) $f_V(G) \subseteq [0, 1]$.
- (ii) $f_V|_K \equiv 1$.
- (iii) $\text{supp } f_V \subseteq U$.
- (iv) $\|f_V\|_{A(G)} \leq (\lambda(KV)/\lambda(V))^{\frac{1}{2}}$.

The existence of $f_V \in A(G) \cap C_c(G)$ is proved in [7, Lemma 3.2], so apparently $f_V \in \mathcal{LA}(G) \cap C_c(G)$ and we have (i), (ii) and (iii). Also based on the proof of [7, Lemma 3.2] and the definition of the norm of $A(G)$, we have (iv).

Lemma 1.2 *Let K be a compact subset of locally compact group G . Let $\{V_1, \dots, V_n\}$ be an open covering of K . Then, there exists $\{f_1, f_2, \dots, f_n\} \subseteq \mathcal{LA}(G) \cap C_c(G)$ such that $\sum_{i=1}^n f_i \equiv 1$ on K and $\text{coz}(f_i) \subseteq V_i$ for $i \in 1, 2, \dots, n$ when $\text{coz}(f_i)$ is the set of all $x \in G$ such that $f_i(x) \neq 0$.*

Proof. for $x \in K$ define U_x a compact neighborhood of x such that $U_x \subseteq V_i$ for some $i \in 1, 2, \dots, n$. Since K is a compact set, there exist $\{x_1, \dots, x_m\} \subset K$ such that $K \subseteq \bigcup_{j=1}^m U_{x_j}$. Let $K_i := \bigcup \{U_{x_j} : U_{x_j} \subseteq V_i\}$ for $i \in 1, 2, \dots, n$. Since each K_i is compact, there exists $g_i \in \mathcal{LA}(G) \cap C_c(G)$ such that $g_i \equiv 1$ on K_i and $\text{coz}(g_i) \subset V_i$, and similarly there is $g \in \mathcal{LA}(G) \cap C_c(G)$ such that $g \equiv 1$ on K , by Lemma 1.1. Also since $\mathcal{LA}(G)$ is an algebra by pointwise multiplication, all of $f_1 := g_1$, $f_2 := (g - g_1) \cdot g_2$, ..., $f_{n-1} := (g - g_1)(g - g_2) \dots (g - g_{n-2}) \cdot g_{n-1}$ and $f_n := (g - g_1)(g - g_2) \dots (g - g_{n-1}) \cdot g_n$ belong to $\mathcal{LA}(G) \cap C_c(G)$. Obviously, $\text{coz}(f_i) \subseteq \text{coz}(g_i) \subseteq V_i$ for $i \in 1, 2, \dots, n$.

Consequently, for each $x \in K$ we will have

$$\begin{aligned} f_1(x) + f_2(x) &= g_1(x) + (1 - g_1(x)) \cdot g_2(x) \\ &= 1 - (1 - g_1(x)) + (1 - g_1(x)) \cdot g_2(x) \\ &= 1 - (1 - g_1(x))(1 - g_2(x)). \end{aligned}$$

By induction we can see that for each $x \in K$

$$f_1(x) + f_2(x) + \dots + f_n(x) = 1 - (1 - g_1(x))(1 - g_2(x)) \dots (1 - g_n(x)).$$

In addition, for each $x \in K$ there exists some $i_0 \in 1, 2, \dots, n$ such that $x \in K_{i_0}$, so $g(x) = g_{i_0}(x) = 1$. As a result $f_1(x) + f_2(x) + \dots + f_n(x) = 1$ for each arbitrary x in K . \square

2 Separating maps on the Lebesgue-Fourier algebras

Definition 2.1 Suppose G_1 and G_2 are two locally compact groups. The linear map $T : \mathcal{LA}(G_1) \rightarrow \mathcal{LA}(G_2)$ is said to be separating or disjointness preserving if $f \cdot g \equiv 0$ implies that $Tf \cdot Tg \equiv 0$.

Let G'_2 be all the $\gamma \in G_2$ for which there exists $f \in \mathcal{LA}(G_1) \cap C_c(G_1)$ such that $Tf(\gamma) \neq 0$. We can easily see that

$$G'_2 = \bigcup \{(Tf)^{-1}(\mathbb{C} \setminus \{0\}) : f \in \mathcal{LA}(G_1) \cap C_c(G_1)\}$$

This shows that G'_2 is an open subset of G_2 . For each $\gamma \in G'_2$, we define $T^t \gamma^t : \mathcal{LA}(G_1) \rightarrow \mathbb{C}$ as $T^t \gamma^t(f) = Tf(\gamma)$ for $f \in \mathcal{LA}(G_1)$. An open subset V of G_1 is called a *vanishing set* for $T^t \gamma^t$ if $T^t \gamma^t(f) = 0$ for all $f \in \mathcal{LA}(G_1) \cap C_c(G)$ that $\text{coz}(f) \subseteq V$.

Proposition 2.2 The set $\text{supp} T^t \gamma^t := G_1 \setminus \bigcup \{V \subset G_1 : V \text{ is a vanishing set for } T^t \gamma^t\}$ is a singleton for each $\gamma \in G'_2$.

Proof. If $\text{supp} T^t \gamma^t$ is empty, we have

$$G_1 = \bigcup \{V \subset G_1 : V \text{ is a vanishing set for } T^t \gamma^t\}.$$

For $f \in \mathcal{LA}(G_1) \cap C_c(G_1)$, let $K = \text{supp} f$ which is a compact set in G_1 . So there exist V_1, V_2, \dots, V_n of vanishing sets for $T^t \gamma^t$ such that $K \subseteq V_1 \cup \dots \cup V_n$. By Lemma 1.2 there are f_1, \dots, f_n in $\mathcal{LA}(G_1)$ such that $\sum_{i=1}^n f_i \equiv 1$ on K and $\text{coz}(f_i) \subseteq V_i$ for $i \in 1, 2, \dots, n$. So $f(x) \equiv \sum_{i=1}^n f \cdot f_i(x)$ on K . On the other hand, each V_i is a vanishing set for $T^t \gamma^t$ and also $\text{coz}(f \cdot f_i) \subseteq \text{coz}(f_i) \subseteq V_i$. Therefore, $T^t \gamma^t(f \cdot f_i) = T(f \cdot f_i)(\gamma) = 0$ for $i \in 1, 2, \dots, n$. Thus, $T(f)(\gamma) = 0$ for all $f \in \mathcal{LA}(G_1) \cap C_c(G_1)$ which is a contradiction according to the definition of G'_2 .

Suppose there are x and y , two different elements in $\text{supp} T^t \gamma^t$. There are V and U two disjoint neighborhoods of x and y , respectively. There exist $f, g \in \mathcal{LA}(G_1) \cap C_c(G)$ such that $T(f)(\gamma) \neq 0$ and $T(g)(\gamma) \neq 0$ while $\text{coz}(f) \subseteq U$ and $\text{coz}(g) \subseteq V$. Moreover $f \cdot g \equiv 0$, since $\text{coz}(f \cdot g) \subseteq \text{coz}(f) \cap \text{coz}(g) \subseteq U \cap V$; meanwhile, $Tf \cdot Tg \neq 0$ which is a contradiction. \square

Definition 2.3 Proposition let us define $t : G'_2 \rightarrow G_1$ when for each $\gamma \in G'_2$, $t(\gamma)$ is the solitary element of $\text{supp} T^t \gamma^t$. We call t the support map of T .

If G is a locally compact group, there exists a one point compactification ωG for G [16]. It is clear that each function in $\mathcal{LA}(G)$, as an element of $C_0(G)$, has a unique extension into $C(\omega G)$.

Proposition 2.4 *The support map t of T is continuous.*

Proof. Let (γ_α) be a net in G'_2 which converges to $\gamma \in G'_2$. If ωG_1 is the one point compactification of G_1 , we know that $(t(\gamma_\alpha)) \subseteq G_1$ has a convergent subnet to some x in ωG_1 , we show this subnet by $(t(\gamma_\beta))$.

Suppose that $t(\gamma) \neq x$. So there are V and U two disjoint neighborhoods of $t(\gamma)$ and x respectively in ωG_1 . Since t is the support map of T , there is $f \in \mathcal{LA}(G_1) \cap C_c(G_1)$ such that $Tf(\gamma) \neq 0$ and $\text{coz}(f) \subseteq V$. But since $Tf \in \mathcal{LA}(G_2)$, it is a continuous function on G'_2 . So there must be β_0 such that $Tf(\gamma_{\beta_0}) \neq 0$ and also $t(\gamma_{\beta_0}) \in U$. There exists $g \in \mathcal{LA}(G_1) \cap C_c(G_1)$ such that $Tg(\gamma_{\beta_0}) \neq 0$ and $\text{coz}(g) \subseteq U$. We have $\text{coz}(f) \cap \text{coz}(g) = \emptyset$ so $f \cdot g \equiv 0$. On the other hand, $Tf \cdot Tg(\gamma_{\beta_0}) \neq 0$ which leads to a contradiction, so $t(\gamma) = x$. \square

Proposition 2.5 *Suppose that U is an open subset of G_1 and $f \in \mathcal{LA}(G_1)$, The following statements are held for t the support map of T .*

- (i) *If $f|_U \equiv 0$, then $Tf|_{t^{-1}(U)} \equiv 0$.*
- (ii) *$t(\text{coz}(Tf) \cap G'_2) \subseteq \text{supp} f$.*

Proof. To prove (i) suppose that $\gamma \in t^{-1}(U)$. Since $t(\gamma) \in U$, there is $g \in \mathcal{LA}(G_1) \cap C_c(G_1)$ such that $\text{coz}(g) \subseteq U$ and $Tg(\gamma) \neq 0$. If $f|_U \equiv 0$ then $\text{coz}(f) \cap \text{coz}(g) = \emptyset$ which implies that $f \cdot g \equiv 0$. The separating property of T shows that $Tf \cdot Tg \equiv 0$ and $Tf \cdot Tg(\gamma) = 0$ therefore $Tf(\gamma) = 0$. If $t^{-1}(U) = \emptyset$, it is trivial.

To prove (ii), if $\gamma \in G'_2$ and $t(\gamma) \notin \text{supp} f$, there is an open neighborhood U of $t(\gamma)$ such that $f|_U \equiv 0$. By (i) we have that $Tf(\gamma) = 0$, so γ does not belong to $\text{coz}(Tf)$. \square

Theorem 2.6 *If T is injective, then $t(G'_2)$ is dense in G_1 .*

Proof. Suppose that there exists $x \in G_1$ which does not belong to $\overline{t(G'_2)}$. So there are U and V two open disjoint subsets of G_1 such that $\overline{t(G'_2)} \subseteq V$ and $x \in U$.

Now let $f \in \mathcal{LA}(G_1) \cap C_c(G_1)$ such that $f(x) \neq 0$ and $\text{coz}(f) \subset U$. Since $\text{coz}(f) \cap t(G'_2) \subseteq U \cap V = \emptyset$, $f|_V \equiv 0$ and by Proposition 2.5 (i), $Tf|_{t^{-1}(V)} \equiv 0$. But we know that $t(G'_2) \subseteq V$; consequently, by the definition of G'_2 we have $Tf \equiv 0$

where $f \neq 0$ which is a contradiction. \square

Theorem 2.7 *If T is onto, $t : G'_2 \rightarrow G_1$ the support map of T has a continuous extension $t^* : G_2 \rightarrow \omega G_1$.*

Proof. Let us define $t^* : G_2 \rightarrow \omega G_1$ such that $t^*(\gamma_0) = \infty$ when $\gamma_0 \in G_2 \setminus G'_2$ and $t^*(\gamma) = t(\gamma)$ when $\gamma \in G'_2$. Since G'_2 is an open subset of G_2 , t^* is continuous on G'_2 . Also, if $\gamma_0 \in G_2 \setminus G'_2$ has a neighborhood V such that $V \subseteq G_2 \setminus G'_2$, t^* would be continuous at γ_0 . Consequently, we suppose that there exists $(\gamma_\alpha) \subseteq G'_2$ that converges to γ_0 ; meanwhile, $(t(\gamma_\alpha))$ converges to some $x \in G_1$.

By Lemma 1.1, there exists $f \in \mathcal{LA}(G_1) \cap C_c(G_1)$ such that $f \equiv 1$ on U , an arbitrary relatively compact neighborhood of x . On the other hand, since T is onto, we can find $g \in \mathcal{LA}(G_1)$ such that $Tg(\gamma_0) \neq 0$, so $g - f \cdot g \equiv 0$ on U and $(T(g - f \cdot g)(\gamma_\alpha))$ converges to $T(g - f \cdot g)(\gamma_0) = Tg(\gamma_0) - T(f \cdot g)(\gamma_0) = Tg(\gamma_0)$, because $f \cdot g \in \mathcal{LA}(G_1) \cap C_c(G_1)$ and $\gamma_0 \notin G'_2$. Therefore, we can find an index α_0 such that $T(g - f \cdot g)(\gamma_{\alpha_0}) \neq 0$ when $t(\gamma_{\alpha_0}) \in U$. By Proposition 2.5 (ii), we have $t(\gamma_{\alpha_0}) \in \text{supp}(g - f \cdot g)$, which is a contradiction because $g - f \cdot g \equiv 0$ on U . So $(t(\gamma_\alpha))$ converges to ∞ and t^* is continuous. \square

Let us define the continuous map $X : t^{-1}(G_1) \rightarrow \mathbb{C}$ as follows: Given $\gamma \in t^{-1}(G_1)$, let U be a relatively compact neighborhood of $t(\gamma)$ and let $f_{\gamma,U}$ be a function in $\mathcal{LA}(G_1) \cap C_c(G_1)$ such that $f_{\gamma,U} \equiv 1$ on U . Then we define $X(\gamma) = Tf_{\gamma,U}(\gamma)$.

For this function to be well-defined, we let V be another relatively compact neighborhood of $t(\gamma)$ and take $f_{\gamma,V}$ as above. By Proposition 2.5 part (i) and since $f_{\gamma,U} - f_{\gamma,V} \equiv 0$ on $V \cap U$, we have $Tf_{\gamma,U} - Tf_{\gamma,V} \equiv 0$ on $t^{-1}(V \cap U)$. Since we have chosen U and V as the neighborhoods of $t(\gamma)$, we have $\gamma \in t^{-1}(V \cap U)$, and it shows that $Tf_{\gamma,U}(\gamma) = Tf_{\gamma,V}(\gamma)$.

To check continuity of X , let (γ_α) be a net in $t^{-1}(G_1)$ that converges to some $\gamma \in t^{-1}(G_1)$; in addition, let U be a relatively compact neighborhood of $t(\gamma)$. As we have seen in Proposition 2.4, t is a continuous function, so $t^{-1}(U)$ is an open neighborhood of γ . But there is α_0 such that for each $\alpha > \alpha_0$, we have $\gamma_\alpha \in t^{-1}(U)$, so $X(\gamma_\alpha) = X(\gamma)$.

Let us consider $\mathcal{LA}(G_1)$ with the topology which is inherited from $(C_0(G_1), \|\cdot\|_\infty)$. We use G''_2 to denote the subset of G'_2 consisting of all $\gamma \in G'_2$ such that $T^t \gamma^t$ is a continuous map from $\mathcal{LA}(G_1)$ with the mentioned topology to \mathbb{C} . G_2^0 denotes the complement of G''_2 in G'_2 .

Theorem 2.8 *For each $\gamma \in G'_2$, $\gamma \in G''_2$ if and only if $Tf(\gamma) = X(\gamma) \cdot f(t(\gamma))$ for all $f \in \mathcal{LA}(G_1)$.*

Proof. First, we will show that if $\gamma \in G''_2$, then $f(t(\gamma)) = 0$ implies that $Tf(\gamma) = 0$ for all $f \in \mathcal{LA}(G)$. Since $f(t(\gamma)) = 0$ and f is a continuous function, we can find an open neighborhood U_n of $t(\gamma)$ for each $n \in \mathbb{N}$ such that

$$\sup\{|f(x)| : x \in \overline{U_n}\} < 1/n.$$

Take a relatively compact neighborhood V_n of $t(\gamma)$ such that $\overline{V_n} \subseteq U_n$ for each $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ we have $g_n \in \mathcal{LA}(G_1) \cap C_c(G_1)$ such that $g_n|_{V_n} \equiv 1$, $g_n(G_1) \subset [0, 1]$, and $\text{supp } g_n \subseteq U_n$.

Since $f \in \mathcal{LA}(G_1) \subseteq C_0(G_1)$ and also following our hypothesis about U_n , by increasing n , $(f \cdot g_n)$ converges to 0 (in the topology which is inherited from $(C_0(G_1), \|\cdot\|_\infty)$ to $\mathcal{LA}(G_1)$) and also $f \cdot g_n \equiv f$ on V_n for each n . By applying Proposition 2.5 part (i) we have $Tf(\gamma) = T(f \cdot g_n)(\gamma)$; consequently, since $\gamma \in G''_2$, $T^t \gamma^t$ is continuous and so

$$Tf(\gamma) = \lim_n T(f \cdot g_n)(\gamma) = 0.$$

Subsequently, suppose that $f(t(\gamma)) \neq 0$, define $g := f - f(t(\gamma)) \cdot f_{\gamma,V}$ where $f_{\gamma,V}$ is constructed as previously described. Since T is a linear map and $g(t(\gamma)) = f(t(\gamma)) - f(t(\gamma)) \cdot f_{\gamma,V}(t(\gamma)) = 0$ because $f_{\gamma,V}(t(\gamma)) = 1$, we have $Tf(\gamma) = X(\gamma) \cdot f(t(\gamma))$ for all $f \in \mathcal{LA}(G_1)$.

Conversely, let (f_n) be a net in $\mathcal{LA}(G_1)$ such that $f_n \rightarrow f$ for some $f \in \mathcal{LA}(G_1)$ in the mentioned topology. Finally, we have

$$\begin{aligned} Tf_n(\gamma) - Tf(\gamma) &= X(\gamma) \cdot f_n(t(\gamma)) - X(\gamma) \cdot f(t(\gamma)) \\ &= X(\gamma) \cdot (f_n(t(\gamma)) - f(t(\gamma))). \end{aligned}$$

□

Proposition 2.9 *G''_2 and G^0_2 have the following properties:*

- (i) G''_2 is a closed subset of G'_2 .
- (ii) $t(G^0_2)$ is a subset of limit points of G_1 .

Proof. To prove (i), let (γ_α) be a net in G''_2 which converges to some $\gamma \in G'_2$. By Theorem 2.8 for each α we have $Tf(\gamma_\alpha) = X(\gamma_\alpha) \cdot f(t(\gamma_\alpha))$ for all $f \in \mathcal{LA}(G_1)$.

Since X and $f \circ t$ are continuous, we have

$$\begin{aligned} \lim_{\alpha} Tf(\gamma_{\alpha}) &= \lim_{\alpha} X(\gamma_{\alpha}) \cdot f(t(\gamma_{\alpha})) \\ &= X(\gamma) \cdot f(t(\gamma)), \end{aligned}$$

for each $f \in \mathcal{LA}(G_1)$. Moreover, since $Tf \in \mathcal{LA}(G_2) \subseteq C_0(G_2)$ and $\lim_{\alpha} Tf(\gamma_{\alpha}) = Tf(\gamma)$, we have $Tf(\gamma) = X(\gamma) \cdot f(t(\gamma))$ for all $f \in \mathcal{LA}(G_1)$, and Theorem 2.8 proves (i).

Suppose that $t(\gamma)$ is an isolated point for G_1 when $\gamma \in G'_2$. For each $f \in \mathcal{LA}(G_1)$ define $\dot{f} := f(t(\gamma)) \cdot f_{\gamma,V}$ when V is a relatively compact neighborhood of γ . We have $f|_{t(\gamma)} = \dot{f}|_{t(\gamma)}$. The singleton $\{t(\gamma)\}$ is an open subset of G_1 , since it is isolated. Proposition 2.5 implies that $Tf(\gamma) = T\dot{f}(\gamma)$, since $X(\gamma) = Tf_{\gamma,V}(\gamma)$ and

$$Tf(\gamma) = T\dot{f}(\gamma) = f(t(\gamma)) \cdot Tf_{\gamma,V}(\gamma)$$

by Theorem 2.8. Consequently, $\gamma \in G''_2$. Therefore, $t(G''_2)$ is a subset of limit points of G_1 . \square

Lemma 2.10 *Let G be a locally compact group and x an element of G with a fixed relatively compact neighborhood U . If f is a function in $A(G)$ so that $f(x) = 0$, then for each $\epsilon > 0$ there exists $V \subset U$ another relatively compact neighborhood of x and $h \in \mathcal{LA}(G)$ such that:*

- (i) $\|h\|_{\infty} \leq 1$ and $\|h\|_{A(G)} \leq 2$.
- (ii) $h|_V \equiv 1$ and $\text{supp} h \subset U$.
- (iii) $\|h \cdot f\|_{A(G)} < \epsilon$.

Proof. Let

$$0 < \delta < \frac{\epsilon}{2\lambda(U)}.$$

Since $f \in A(G)$, it belongs to $C_0(G)$. Now define $B_{\delta} := \{z \in \mathbb{C} : |z| < \delta\}$, so $W := f^{-1}(B_{\delta}) \cap U$ is a relatively compact open subset of G which contains x and $W \subseteq U$. Moreover, there exists O , a relatively compact neighborhood of e , such that $xOO \subseteq W$. Since λ is a Radon measure we can find V , a relatively compact neighborhood of x , such that $\lambda(xO) \leq \lambda(VO) \leq 2\lambda(xO)$; meanwhile, $VOO \subseteq W$.

By Lemma 1.1, there exists some $k \in \mathcal{LA}(G)$, such that $k(G) \subseteq [0, 1]$, $k|_V \equiv 1$, $\text{supp} k \subseteq W$, and $\|k\|_{A(G)} \leq \sqrt{2}$.

Let us consider $k \cdot f$ a member of $C_c(G)$ such that $\text{supp} k \cdot f \subseteq W$, $\|k \cdot f\|_{\infty} \leq \delta$,

and $k \cdot (k \cdot f)$ is a result of a $C^*(G)$ -action on $B(G)$ (as $C^*(G)$ is predual of $B(G)$ and $A(G) \subset B(G)$ and their norms are identity on $A(G)$). So we can write

$$\|k \cdot k \cdot f\|_{A(G)} \leq \|k\|_{A(G)} \|k \cdot f\|_{\Sigma}$$

where $\|\cdot\|_{\Sigma}$ shows the norm of $C^*(G)$ and we have $\|\cdot\|_{\Sigma} \leq \|\cdot\|_1$ (see [7]). If we define h as $k \cdot k$ which belongs to $\mathcal{LA}(G)$, it is clear that h would satisfies (i) and (ii). Finally, since $\text{supp}(k \cdot f) \subset W$, for (iii) we have

$$\begin{aligned} \|h \cdot f\|_{A(G)} &= \|k \cdot k \cdot f\|_{A(G)} \\ &\leq \|k\|_{A(G)} \|k \cdot f\|_{\Sigma} \\ &\leq \|k\|_{A(G)} \|k \cdot f\|_1 \\ &\leq \sqrt{2}\lambda(W) \|k \cdot f\|_{\infty} \\ &< 2\lambda(U) \delta < \epsilon \end{aligned}$$

□

Theorem 2.11 $t(G_2^0) \cap K^o$ is finite for every compact subset K of G_1 , where K^o denotes the interior of K .

Proof. To obtain a contradiction suppose that there is a sequence of distinct elements of G_2^0 as (γ_n) such that $(t(\gamma_n)) \subseteq K^o$ for some compact set K in G_1 . Therefore, we can assume that (U_n) is a pairwise disjoint sequence of open subsets of K^o such that $t(\gamma_n) \in U_n$ for each $n \in \mathbb{N}$.

For each n , $\gamma_n \in G_2^0$, so there exists $g_n \in \mathcal{LA}(G_1)$ such that $Tg_n(\gamma_n) \neq X(\gamma_n) \cdot g_n(t(\gamma_n))$. Let $f_n := g_n - (g_n(t(\gamma_n))) \cdot f_{\gamma_n, V}$ where V is a relatively compact neighborhood of K . It is clear that for each n , f_n is also a member of $\mathcal{LA}(G_1)$. By our assumption about g_n , it is obvious that $Tf_n(\gamma_n) \neq 0$ and $f_n(t(\gamma_n)) = 0$. Since T is linear, we may assume that $|Tf_n(\gamma_n)| > n$ for each $n \in \mathbb{N}$.

By Lemma 2.10, for $n \in \mathbb{N}$ we can find $V_n \subseteq U_n$ a neighborhood of $t(\gamma_n)$ and $h_n \in \mathcal{LA}(G_1)$ such that $\|h_n\|_{A(G_1)} \leq 2$, $h_n|_{V_n} \equiv 1$, $\text{supp} h_n \subset U_n$, and $\|h_n \cdot f_n\|_{A(G_1)} < \frac{1}{2^n}$.

Now we can define $y_n := h_n \cdot f_n$ for each $n \in \mathbb{N}$. Since $h_n|_{V_n} \equiv 1$, $(y_n - f_n)|_{V_n} \equiv 0$, and

$$|Ty_n(\gamma_n)| = |Tf_n(\gamma_n)| > n$$

for each $n \in \mathbb{N}$, by Proposition 2.5. Consequently, for each $n \in \mathbb{N}$ we can say

$$\|y_n\|_1 = \|h_n \cdot f_n\|_1$$

$$\begin{aligned}
&\leq \lambda(U_n) \|h_n \cdot f_n\|_\infty \\
&\leq \lambda(V) \|h_n \cdot f_n\|_{A(G_1)} \\
&< \lambda(V) \frac{1}{2^n}.
\end{aligned}$$

So it is apparent that $y := \sum_n y_n$ not only is an element of $A(G)$ but also belongs to $\mathcal{LA}(G)$. Based on our assumption about (U_n) and since $\text{supp } y_n \subset U_n$, we have $y_n|_{U_m} \equiv 0$ for each $m \neq n$. By Proposition 2.5, we have $Ty_n|_{t^{-1}(U_m)} \equiv 0$ for $m \neq n$. Therefore, $|Ty(\gamma_n)| = |Ty_n(\gamma_n)| > n$ for each n , and it leads to unboundedness of Ty which is a contradiction. \square

Proposition 2.12 *The mapping X defined above is bounded on G'_2 .*

Proof. If X is not bounded there is a sequence (γ_n) in G''_2 such that $|X(\gamma_n)| > 4^n$ for each $n \in \mathbb{N}$. If $(t(\gamma_n))$ was a finite set in G_1 , we can assume that $t(\gamma_n) = x$ for each $n \in \mathbb{N}$. There is $f \in \mathcal{LA}(G_1)$ such that $f(x) = 1$. Since $\gamma_n \in G''_2$, we have

$$|Tf(\gamma_n)| = |X(\gamma_n) \cdot f(t(\gamma_n))| = |X(\gamma_n)| \cdot |f(x)| > 4^n$$

which is contradictory.

On the other hand, if $(t(\gamma_n))$ is infinite, we can assume that $t(\gamma_n) \neq t(\gamma_m)$ when $n \neq m$. Let (U_n) be a sequence of pairwise disjoint relatively compact open subsets of G_1 such that $t(\gamma_n) \in U_n$ for each $n \in \mathbb{N}$. Fix U_0 a relatively compact neighborhood of $e \in G_2$. Define $V_n = U_n \cap t(\gamma_n)U_0$ for each $n \in \mathbb{N}$.

By Lemma 2.10, there exists sequence (h_n) in $\mathcal{LA}(G_1)$ such that $\text{supp } h_n \subseteq V_n$, $h_n(t(\gamma_n)) = 1$, $\|h_n\|_\infty \leq 1$, and $\|h_n\|_{A(G_1)} \leq 2$ for each n . Define $y_n := h_n/2^n$, so $y_n(t(\gamma_n)) = 1/2^n$. Also we have $\|y_n\|_\infty \leq 1/2^n$.

Consequently, we define $y := \sum_n y_n$ which is in $A(G_1)$. Moreover, since

$$\|y_n\|_1 \leq \lambda(V_n) \|y_n\|_\infty \leq \frac{1}{2^n} \lambda(U_0),$$

y is a member of $L^1(G_1)$; thus, $y \in \mathcal{LA}(G_1)$. Since (V_n) are pairwise disjoint open neighborhoods, $y|_{V_n} \equiv y_n|_{V_n}$ for each n . Eventually, by Proposition 2.5, we have

$$|Ty(\gamma_n)| = |Ty_n(\gamma_n)| = |X(\gamma_n) \cdot y_n(t(\gamma_n))| > 4^n \cdot 1/2^n = 2^n,$$

which is a contradiction since $Ty \in \mathcal{LA}(G_2) \subseteq C_0(G_2)$. \square

3 Automatic continuity of bijective separating maps

Theorem 3.1 *Let T be a bijective separating map of $\mathcal{LA}(G_1)$ onto $\mathcal{LA}(G_2)$. Then $G_2'' = G_2$ and T is continuous.*

Moreover, t is an injective homeomorphism from G_2 onto G_1 .

Proof. By Theorem 2.6, since T is bijective, $t(G_2')$ is a dense subset of G_1 . Moreover, by Theorem 2.11, for an arbitrary element $x \in G_1$ and a relatively compact neighborhood U of x , $U \cap t(G_2^0)$ has finite number of elements. Also by Proposition 2.9, we know that $t(G_2^0)$ is a subset of limit points of G_1 .

If G_1 is discrete, since $t(G_2')$ is dense in G_1 , we have $t(G_2') = G_1$ and $U \cap t(G_2') \neq \emptyset$. Otherwise U has infinite number of elements. Since $U \cap t(G_2^0)$ has finite number of elements, $U \cap t(G_2') \neq \emptyset$. We can conclude that $t(G_2'')$ is dense in G_1 .

On the other hand, if $\gamma \in G_2''$ then $Tf(\gamma) = X(\gamma) \cdot f(t(\gamma))$ for each $f \in \mathcal{LA}(G_1)$. Since T is onto and $\mathcal{LA}(G_2)$ is a separating subalgebra of $C_0(G)$, $X(\gamma) \neq 0$ for each $\gamma \in G_2''$. So if for $f \in \mathcal{LA}(G_1)$ we have $Tf(\gamma) = 0$ where $\gamma \in G_2''$, it implies that $f(t(\gamma)) = 0$. Since $t(G_2'')$ is dense in G_1 , if $Tf|_{G_2''} \equiv 0$ then $f \equiv 0$ on G_1 .

Toward a contradiction suppose that $\gamma \in G_2' \setminus G_2''$. By Proposition 2.9 we know that G_2'' is closed in G_2' . So there exists a closed subset C of G_2 such that $G_2'' = G_2' \cap C$. Apparently, $\gamma \notin C$; thus, by Lemma 1.1 and since T is onto, there exists $f \in \mathcal{LA}(G_1)$ such that $Tf|_C \equiv 0$ and $Tf(\gamma) = 1$. Therefore $Tf|_{G_2''} \equiv 0$ and $f \equiv 0$, which makes a contradiction. Consequently, $G_2'' = G_2'$.

To show that $G_2 = G_2'$, we first show that $G_2 \setminus G_2'$ is open in G_2 . Let $\gamma_0 \in G_2 \setminus G_2'$ and $(\gamma_\alpha) \subseteq G_2'$ be a net which converges to γ_0 . Based on the proof of Theorem 2.7, $(t(\gamma_\alpha))$ converges to ∞ in ωG_1 . Since T is onto, there exists $f \in \mathcal{LA}(G_1)$ such that $Tf(\gamma_0) \neq 0$. Hence, since X is bounded on G_2' and $Tf(\gamma_\alpha) = X(\gamma_\alpha) \cdot f(t(\gamma_\alpha))$ for each α , $(f(t(\gamma_\alpha)))$ converges to 0. Because Tf is continuous, convergence of $(f(t(\gamma_\alpha)))$ implies that $Tf(\gamma_0) = 0$ which is a contradiction, so $G_2 \setminus G_2'$ is open.

Now suppose that $G_2 \setminus G_2' \neq \emptyset$. Since $G_2 \setminus G_2'$ is open, there exists $g \in \mathcal{LA}(G_2)$ such that $g \neq 0$ and $\text{coz}(g) \subseteq G_2 \setminus G_2'$. Since T is onto, there exists $f \in \mathcal{LA}(G_1)$ such that $Tf = g$. Let $x \in t(G_2')$ and $\gamma \in G_2'$ such that $t(\gamma) = x$. Thus $0 = g(\gamma) = Tf(\gamma) = X(\gamma) \cdot f(t(\gamma))$ which implies that $f(x) = f(t(\gamma)) = 0$, since $X(\gamma) \neq 0$. So $f|_{t(G_2')} \equiv 0$ and $t(G_2')$ is dense in G_1 ; consequently, $f \equiv 0$ which is impossible. Therefore $G_2 = G_2'$ and $G_2 = G_2''$, and by Theorem 2.8, $Tf(\gamma) = X(\gamma) \cdot f(t(\gamma))$ for each $\gamma \in G_2$ and $f \in \mathcal{LA}(G_1)$.

Now we will show that t is an injective map. Let γ_1 and γ_2 be two distinct elements of G_2 . Suppose that $t(\gamma_1) = t(\gamma_2)$. Since $Tf(\gamma_1) = X(\gamma_1) \cdot f(t(\gamma_1))$ and $Tf(\gamma_2) = X(\gamma_2) \cdot f(t(\gamma_2))$ for each $f \in \mathcal{LA}(G_1)$, and the map $X(\gamma) \neq 0$ for each $\gamma \in G_2$, we deduce that $Tf(\gamma_1) = (X(\gamma_1)/X(\gamma_2)) \cdot Tf(\gamma_2)$. Also T is onto, so we can

find $f_0 \in \mathcal{LA}(G_1)$ such that $Tf_0(\gamma_1) = 1$ and $Tf_0(\gamma_2) = 0$, which is a contradiction, so t should be an injective map from G_2 to G_1 .

To show that T is a continuous map, we use the closed graph theorem. Suppose that (f_n) is a sequence in $\mathcal{LA}(G_1)$ which converges to some $f \in \mathcal{LA}(G_1)$. Also suppose that (Tf_n) converges to some $g \in \mathcal{LA}(G_2)$. By the definition of the norm of the Lebesgue-Fourier algebra, we know that $\|\cdot\|_\infty \leq \|\cdot\|_{A(G)} \leq |||\cdot|||$. So $|||f_n - f||| \rightarrow 0$ shows that $\|f_n - f\|_\infty \rightarrow 0$. Based on the definition of G_2'' and since $G_2 = G_2''$, for each $\gamma \in G_2$ we have $|T^t \gamma^t(f_n) - T^t \gamma^t(f)| \rightarrow 0$ which implies that $|Tf_n(\gamma) - Tf(\gamma)| \rightarrow 0$ for each $\gamma \in G_2$. Since $|||Tf_n - g||| \rightarrow 0$, $\|Tf_n - g\|_\infty \rightarrow 0$ and hence for each $\gamma \in G_2$, $Tf(\gamma) = g(\gamma)$ which ends this part of the proof.

To prove the last part, it is necessary to prove that the inverse map of T is also a separating map. Suppose that $g_1, g_2 \in \mathcal{LA}(G_2)$ such that $\text{coz}(g_1) \cap \text{coz}(g_2) = \emptyset$. There exist $f_1, f_2 \in \mathcal{LA}(G_1)$ such that $Tf_1 = g_1$ and $Tf_2 = g_2$. Since $\mathcal{LA}(G_2)$ separates the points of G_2 and also T is onto, $X(\gamma) \neq 0$ for each $\gamma \in G_2$. Now suppose that $t(\gamma) \in \text{coz}(f_1) \cap \text{coz}(f_2)$ for some $\gamma \in G_2$. It implies that

$$\begin{aligned} g_1(\gamma) \cdot g_2(\gamma) &= Tf_1(\gamma) \cdot Tf_2(\gamma) \\ &= X(\gamma) \cdot (f_1(t(\gamma)) \cdot f_2(t(\gamma))) \neq 0 \end{aligned}$$

that is impossible. Since $t(G_2)$ is a dense subset of G_1 and f_1 and f_2 are continuous, $\text{coz}(f_1) \cap \text{coz}(f_2) = \emptyset$, so T^{-1} is a separating map. Since $K := T^{-1}$ is a separating map, we can define $k : G_1 \rightarrow G_2$ its support map as in Definition 2.3. It is clear that k is continuous, injective and $k(G_1)$ is dense in G_2 .

Let us prove $t \circ k(x) = x$ for every $x \in G_1$. Suppose that $t(k(x)) \neq x$ for some $x \in G_1$. So there exist V and U two disjoint relatively compact neighborhoods of x and $t(k(x))$, respectively. By the definition of t , there exists $f_0 \in \mathcal{LA}(G_1)$ such that $\text{supp} f_0 \subseteq U$ and $Tf_0(k(x)) \neq 0$. By Proposition 2.5 (ii), we know that $t(k(x)) \in \text{supp} f_0$ and $x \notin \text{supp} f_0$. Let $f_1 \in \mathcal{LA}(G_1)$ such that $f_1(x) \neq 0$ and $f_1|_U \equiv 0$, so $\text{coz}(f_1) \cap \text{coz}(f_0) = \emptyset$. Since T is separating, we have that $\text{coz}(Tf_1) \cap \text{coz}(Tf_0) = \emptyset$.

On the other hand, since $Tf_0 \in \mathcal{LA}(G_2) \subseteq C_0(G_2)$,

$$\text{coz}(Tf_0) = \{\gamma : Tf(\gamma) \in \mathbb{C} \setminus \{0\}\}$$

which is an open neighborhood of $k(x)$. But for each $g \in \mathcal{LA}(G_2)$ when $\text{coz}(g) \subset \text{coz}(Tf_0)$, we have $\text{coz}(g) \cap \text{coz}(Tf_1) = \emptyset$. Also $\text{coz}(Kg) \cap \text{coz}(f_0) = \emptyset$, since K is a separating map and $KTf_1 = f_1$. So $Kg(x) = 0$ for each $g \in \mathcal{LA}(G_2)$ which is impossible.

Thus $t(k(x)) = x$ for each $x \in G_1$, and since k and t are both continuous, we have that $t^{-1} = k$, and consequently, t is a homeomorphism between G_1 and G_2 . \square

Lemma 3.2 *If T is a bijective separating map, there exists $r > 0$ such that $X(\gamma) > r$ for each $\gamma \in G_2''$.*

Proof. Suppose that $(\gamma_n) \subset G_2''$ is a distinct sequence such that $|X(\gamma_n)| < 1/4^n$ for each $n \in \mathbb{N}$. Let us fix U_0 a relatively compact neighborhood of $e \in G_2$. We can find (U_n) a pairwise disjoint sequence of open sets such that U_n is a neighborhood of γ_n and $\gamma_n^{-1}U_n \subseteq U_0$ for each n . By Lemma 2.10, there exists $g_n \in \mathcal{LA}(G_2)$ such that $g_n(\gamma_n) = 1/3^n$, $\text{supp } g_n \subseteq U_n$, and $\|g_n\|_{A(G_2)} \leq 2/3^n$ for each $n \in \mathbb{N}$. Let us define $g := \sum_{n \in \mathbb{N}} g_n$, we can see that g is an element of $\mathcal{LA}(G_2)$. Since T is a bijection, there exists $f \in \mathcal{LA}(G_1)$ such that $Tf = g$. Hence $|f(t(\gamma_n))| > 4^n/3^n$, since $|Tf(\gamma_n)| = |g(\gamma_n)| = |X(\gamma_n)| \cdot |f(t(\gamma_n))|$ for each $n \in \mathbb{N}$. But as we have seen in Theorem 3.3, t is injective, so it leads to a contradiction with respect to the boundedness of f . \square

Therefore, we can summarize the previous results in the following theorem.

Theorem 3.3 *Let T be a bijective separating map of $\mathcal{LA}(G_1)$ onto $\mathcal{LA}(G_2)$. Then T is continuous.*

Moreover, $Tf(\gamma) = X(\gamma) \cdot f(t(\gamma))$ for all $f \in \mathcal{LA}(G_1)$ and $\gamma \in G_2$, where t is an injective homeomorphism from G_2 into G_1 and X is a bounded continuous function on G_2 which is bounded away from 0.

This theorem helps us to extend every bijective separating map to the Fourier algebras of locally compact groups:

Corollary 3.4 *If $T : \mathcal{LA}(G_1) \rightarrow \mathcal{LA}(G_2)$ be a bijective separating map, then T has a unique continuous extension to a separating bijective map $\dot{T} : A(G_1) \rightarrow A(G_2)$.*

Proof. Since $T : (\mathcal{LA}(G_1), ||| \cdot |||) \rightarrow (\mathcal{LA}(G_2), ||| \cdot |||)$ is a continuous linear map and $\|\cdot\|_{A(G)} \leq ||| \cdot |||$, by using closed graph theorem we can see that

$$T : (\mathcal{LA}(G_1), \|\cdot\|_{A(G_1)}) \rightarrow (\mathcal{LA}(G_2), \|\cdot\|_{A(G_2)})$$

is a continuous linear map. We know that $\mathcal{LA}(G)$ is a dense ideal in $A(G)$, [12], so we have a continuous linear extension

$$\dot{T} : (A(G_1), \|\cdot\|_{A(G_1)}) \rightarrow (A(G_2), \|\cdot\|_{A(G_2)}).$$

The density of $\mathcal{LA}(G_1)$ in $A(G_1)$ implies the uniqueness of \dot{T} .

For each $f \in A(G_1)$ we have a sequence (f_n) in $\mathcal{LA}(G_1)$ that converges to f .

But by Theorem 3.3, for each $\gamma \in G_2$, we have $Tf_n(\gamma) = X(\gamma) \cdot f_n(t(\gamma))$. Also since \dot{T} is continuous, we have $Tf_n \rightarrow \dot{T}f$ in $\|\cdot\|_{A(G_2)}$. We know that $\|\cdot\|_\infty \leq \|\cdot\|_{A(G_2)}$, and it leads to $Tf_n(\gamma) \rightarrow \dot{T}f(\gamma)$. Similar reasoning shows that $f_n(t(\gamma)) \rightarrow f(t(\gamma))$ which eventually implies that $\dot{T}f(\gamma) = X(\gamma) \cdot f(t(\gamma))$. Theorem 3.3 shows that t is a homeomorphism between G_1 and G_2 and X is a non vanishing function on G_2 , so for each non zero function $f \in A(G_1)$ we have $\dot{T}f \neq 0$ and \dot{T} is injective.

To see that \dot{T} is onto, suppose that $g \in A(G_2)$. There exists a sequence (g_n) in $\mathcal{LA}(G_2)$ that converges to g . Since T is onto, there is (f_n) in $\mathcal{LA}(G_2)$ such that $Tf_n = g_n$ for each $n \in \mathbb{N}$. Therefore we have $g_n(\gamma) = X(\gamma) \cdot f_n(t(\gamma))$ for each $\gamma \in G_2$ and $n \in \mathbb{N}$. Since $g_n = X \cdot f_n \circ t$ and (g_n) is a Cauchy sequence in $A(G_2)$, we have that (f_n) is also a pointwise Cauchy sequence on G_1 by Lemma 3.2, so there is $f \in A(G_1)$ which (f_n) converges to f . Clearly $\dot{T}f = g$. \square

A similar procedure will be used to extend T on $C_0(G_1)$. The proof has two parts, at first by using Corollary 3.4 we extend T on $A(G_1)$ and then by a similar approach we obtain the final extension (see the proof of [10, Theorem 1]). Moreover, based on [2, Theorem 3.8], we can characterize the extension of T on $C_0(G_1)$; even more, [2, Proposition 3.3] asserts the uniqueness of this decomposition.

Corollary 3.5 *Let $T : \mathcal{LA}(G_1) \rightarrow \mathcal{LA}(G_2)$ be a bijective separating map, then T has a unique continuous extension to a separating bijective map $\dot{T} : C_0(G_1) \rightarrow C_0(G_2)$. Also there exists a unique isomorphism $\Phi : C_0(G_1) \rightarrow C_0(G_2)$ and a unique element h in $C_b(G_2)$ such that $T(f) = h \cdot \Phi(f)$ for each $f \in C_0(G_1)$. Moreover, $h(\gamma) \neq 0$ for each $\gamma \in G_2$, $\|h\|_\infty \leq \|T\|$, and $\|\Phi^{-1}\| \leq \|T\| \cdot \|T^{-1}\|$ when Φ^{-1} and T^{-1} denote the inverse of Φ and T respectively.*

4 Algebraic characterization of locally compact groups

Since $\mathcal{LA}(G)$ is a Banach algebra by convolution, we can obtain similar results to [9, Section 5] for Lebesgue-Fourier algebras. The results would be a promotion for t as an injective homeomorphism to a topological group isomorphism from G_2 onto G_1 , under a specified condition, (P) , that is defined as follows. In this section $f * g$ denotes the convolution product of $f, g \in \mathcal{LA}(G)$ inherited from $L^1(G)$ (see [13]).

Definition 4.1 *A linear operator $T : \mathcal{LA}(G_1) \rightarrow \mathcal{LA}(G_2)$ satisfies condition (P) if for each $f, g \in \mathcal{LA}(G_1)$ and $\gamma \in G_2$ such that $T(f * g)(\gamma) = 0$, we have $Tf * Tg(\gamma) = 0$.*

Lemma 4.2 *Let $T : \mathcal{LA}(G_1) \rightarrow \mathcal{LA}(G_2)$ be a map such that $Tf = X \cdot f \circ t$ for each $f \in \mathcal{LA}(G_1)$ where X is a non vanishing scalar valued continuous function defined on G_2 and t is a homeomorphism map from G_2 onto G_1 . If T satisfies condition (P), then t is a group homomorphism.*

Proof. The main idea of proof is similar with the one in [9, Lemma 2]. We will mention the outline of the proof and omit the details. Let $\gamma_1, \gamma_2 \in G_2$ such that $t(\gamma_1\gamma_2) \neq t(\gamma_1)t(\gamma_2)$. So there are open neighborhoods U and V , respectively, for $t(\gamma_1)$ and $t(\gamma_2)$ such that $t(\gamma_1\gamma_2) \notin UV$. Based on Lemma 1.1, we can find $f, g \in \mathcal{LA}(G_1)$ such that $f, g(G_1) \subseteq [0, 1]$, $f(t(\gamma_1)) = g(t(\gamma_2)) = 1$, $\text{supp} f \subseteq U$, and $\text{supp} g \subseteq V$.

First suppose that X is a real valued function on G_2 and $X(\gamma_1) > 0$ and $X(\gamma_2) < 0$. So there is a relatively compact neighborhood W of $e \in G_2$ such that $X|_{\gamma_1 W} > 0$ and $X|_{W\gamma_2} < 0$. If we define $r_x h(y) = h(xy)$ for $x, y \in G_1$ and $h \in \mathcal{LA}(G_1)$, it is clear that $r_x h \in \mathcal{LA}(G_1)$ for each $h \in \mathcal{LA}(G_1)$ (see [7]). We have

$$\int_{\gamma_1 W} X(y)X(y^{-1}\gamma_1\gamma_2)(f \circ t)(y)(r_{t(\gamma_1\gamma_2)}g \circ t)(y^{-1})dy < 0.$$

Since t is a homeomorphism, $t(\gamma_1 W)$ is an open subset of G_1 . Again by using Lemma 1.1, let $k_1, k_2 \in \mathcal{LA}(G_1)$ such that $k_1, k_2(G_1) \subseteq [0, 1]$, $k_1(t(\gamma_1)) = k_2(t(\gamma_2)) = 1$, $\text{supp} k_1 \subseteq t(\gamma_1 W)$, and $\text{supp} k_2 \subseteq t(\gamma_2 W)$. So we can see that $((k_1 \cdot f) * (k_2 \cdot g))(t(\gamma_1\gamma_2)) = 0$. Since the characterization of T on $(k_1 \cdot f) * (k_2 \cdot g)$ is known, we have $T((k_1 \cdot f) * (k_2 \cdot g))(\gamma_1\gamma_2) = 0$.

On the other hand, since T satisfies (P), $(T(k_1 \cdot f) * T(k_2 \cdot g))(\gamma_1\gamma_2) = 0$. But $(T(k_1 \cdot f) * T(k_2 \cdot g))(\gamma_1\gamma_2)$ is equal to

$$\begin{aligned} \int_{G_2} X(y)X(y^{-1}\gamma_1\gamma_2)(k_1(t(y)))(k_2(t(y^{-1}\gamma_1\gamma_2)))(f \circ t)(y)(r_{t(\gamma_1\gamma_2)}g \circ t)(y^{-1}) = \\ \int_{\gamma_1 W} X(y)X(y^{-1}\gamma_1\gamma_2)(k_1(t(y)))(k_2(t(y^{-1}\gamma_1\gamma_2)))(f \circ t)(y)(r_{t(\gamma_1\gamma_2)}g \circ t)(y^{-1}) \end{aligned}$$

which is not zero, and it is a contradiction. Similar to the proof of [9, Lemma 2], we can extend this proof to an arbitrary X as a complex valued function on G_2 and it finishes the proof. \square

By Lemma 4.2 and Theorem 3.3, we obtain the following theorem.

Theorem 4.3 *Let T be a separating bijection of $\mathcal{LA}(G_1)$ onto $\mathcal{LA}(G_2)$. If T satisfies condition (P), its support map t is a topological group isomorphism from G_2 onto G_1 , i.e. it is a topological homeomorphism that meanwhile acts as a group isomorphism.*

5 Characterization of bijective separating maps between Lebesgue-Fourier algebras on amenable locally compact groups

In this section we suppose that G_1 and G_2 are amenable locally compact groups. To pursue our discussion we need the following lemma. This lemma is a generalization of [17, Theorem 3.8.1], an extension from locally compact abelian groups to amenable locally compact groups as well as from $A(G)$ to $\mathcal{LA}(G)$. Derighetti in [5] has proved it for Fourier algebra of an amenable locally compact group. It is straightforward to modify the proof to prove the following lemma.

Lemma 5.1 *Let G be an amenable locally compact group and h be a function defined on G such that $h \cdot f \in B(G)$ for every $f \in \mathcal{LA}(G)$. Then $h \in B(G)$.*

Theorem 5.2 *Let G_1 and G_2 be two amenable locally compact groups. T is a bijective separating map if and only if $T = T_2 \circ T_1$ when $T_1 : \mathcal{LA}(G_1) \rightarrow \mathcal{LA}(G_2)$ is an algebra isomorphism and $T_2 : \mathcal{LA}(G_2) \rightarrow \mathcal{LA}(G_2)$ is a continuous linear transformation in the form of $T_2(g) = g \cdot h$ for all $g \in \mathcal{LA}(G_2)$ for some fixed $h \in B(G_2)$.*

Proof. First, suppose that T is a bijective separating map. We show that X is an element of $B(G_2)$. We use [7, Lemma 2.25.]: let $\alpha_1, \dots, \alpha_n$ be constants in \mathbb{C} and $\gamma_1, \dots, \gamma_n$ be elements in G_2 such that $\|\sum_{i=1}^n \alpha_i \delta_{\gamma_i}\|_{\Sigma} \leq 1$.

Consider $K := \{t(\gamma_1), \dots, t(\gamma_n)\}$ as a compact set in G_1 and $\epsilon > 0$ arbitrary. By [6] and since G_1 is amenable, we have V , a neighborhood of e , such that $\lambda(KV) \leq (1 + \epsilon)^2 \lambda(V)$. Based on Lemma 1.1, we know that there exists $f \in \mathcal{LA}(G_1)$ such that $f(t(\gamma_i)) = 1$ for each $i \in 1, \dots, n$ and $\|f\|_{A(G_1)} \leq 1 + \epsilon$. If we denote Tf by g , we can write

$$\begin{aligned}
 \left| \sum_{i=1}^n \alpha_i X(\gamma_i) \right| &= \left| \sum_{i=1}^n \alpha_i X(\gamma_i) \cdot f(t(\gamma_i)) \right| \\
 &= \left| \sum_{i=1}^n \alpha_i Tf(\gamma_i) \right| \\
 &= \left| \sum_{i=1}^n \alpha_i g(\gamma_i) \right| \\
 &= \left| \sum_{i=1}^n \alpha_i \langle \delta_{\gamma_i}, g \rangle \right| \\
 &= \left| \left\langle \sum_{i=1}^n \alpha_i \delta_{\gamma_i}, g \right\rangle \right|
 \end{aligned}$$

$$\begin{aligned}
&\leq \|g\|_{A(G_2)} \left\| \sum_{i=1}^n \alpha_i \delta_{\gamma_i} \right\|_{\Sigma} \\
&\leq \|Tf\|_{A(G_2)} \\
&\leq \|T\|(1 + \epsilon).
\end{aligned}$$

Therefore

$$\left| \sum_{i=1}^n \alpha_i X(\gamma_i) \right| \leq \|T\|.$$

So based on [7, Lemma 2.25.], we have $X \in B(G_2)$.

Subsequently, we will prove that $f \circ t \in \mathcal{LA}(G_2)$ for all $f \in \mathcal{LA}(G_1)$. Let $\gamma \in G_2$ and $g_\gamma \in \mathcal{LA}(G_2)$ such that $g_\gamma(\gamma) = 1$. As we have seen in the proof of Theorem 3.3, $K := T^{-1}$ is a bijective separating map, so $Kg(x) = Y(x) \cdot g(k(x))$ for all $g \in \mathcal{LA}(G_2)$ and $x \in G_1$ when $Y : G_1 \rightarrow \mathbb{C}$ is a continuous map defined on G_1 similar to X . Now we have

$$\begin{aligned}
1 = g_\gamma(\gamma) &= T(Kg_\gamma)(\gamma) \\
&= X(\gamma) \cdot Kg_\gamma(t(\gamma)) \\
&= X(\gamma) \cdot Y(t(\gamma)) \cdot g_\gamma(k(t(\gamma))) \\
&= X(\gamma) \cdot Y(t(\gamma))
\end{aligned}$$

which shows that $X(\gamma) \cdot Y(t(\gamma)) = 1$ for each $\gamma \in G_2$.

We know that $Y \in B(G_1)$ and $Y \cdot (g \circ k) = Kg \in \mathcal{LA}(G_1)$. Since $A(G_1)$ is an ideal in $B(G_1)$ and $B(G_1) \subseteq C_b(G_1)$, we can see that $\mathcal{LA}(G_1)$ is an ideal in $B(G_1)$; consequently, $Y \cdot Y \cdot (g \circ k) \in \mathcal{LA}(G_1)$. Now we can consider

$$\begin{aligned}
T(Y \cdot Y \cdot (g \circ k))(\gamma) &= X(\gamma) \cdot Y(t(\gamma)) \cdot Y(t(\gamma)) \cdot g(k(t(\gamma))) \\
&= Y(t(\gamma)) \cdot g(\gamma)
\end{aligned}$$

for an arbitrary $\gamma \in G_2$. This implies that $(Y \circ t) \cdot g$ is a function in $\mathcal{LA}(G_2)$ for all $g \in \mathcal{LA}(G_2)$. Lemma 5.1 shows that $Y \circ t$ is a function in $B(G_2)$. Eventually, since $\mathcal{LA}(G_2)$ is an ideal in $B(G_2)$ and $X \cdot (f \circ t) \in \mathcal{LA}(G_2)$, we have that $(Y \circ t) \cdot X \cdot (f \circ t) = f \circ t$ belongs to $\mathcal{LA}(G_2)$ for all $f \in \mathcal{LA}(G_1)$.

Now let us define $T_1 : \mathcal{LA}(G_1) \rightarrow \mathcal{LA}(G_2)$ when $T_1 f = f \circ t$ for each $f \in \mathcal{LA}(G_1)$, and $T_2 : \mathcal{LA}(G_2) \rightarrow \mathcal{LA}(G_2)$ when $Tg = X \cdot g$ for each $g \in \mathcal{LA}(G_2)$. It is obvious that T_2 is a linear transformation on $\mathcal{LA}(G_2)$, and it is bounded by the norm of X which is finite. So T_2 is a continuous linear transformation.

T_1 is an injective algebra homomorphism. Let us suppose that it is not onto, so there exists $g \in \mathcal{LA}(G_2)$ such that $T_1 f \neq g$ for each $f \in \mathcal{LA}(G_1)$. Since T_2 is defined on $\mathcal{LA}(G_2)$, we can write $Tf = T_2(T_1 f) \neq T_2 g$ for all $f \in \mathcal{LA}(G_1)$ which is

impossible.

The converse is trivial. \square

Indeed when G_1 and G_2 are amenable locally compact groups we can say that each bijective separating map is a weighted isomorphism, see [2, Section 3.1]. To prove that this terminology is correct for Theorem 5.2, we just observe that $B(G)$ indeed is the set of all centralizers of $\mathcal{LA}(G)$ for each amenable locally compact group.

Corollary 5.3 *Let G_1 and G_2 be two amenable locally compact groups. If $T : \mathcal{LA}(G_1) \rightarrow \mathcal{LA}(G_2)$ is a bijective separating map, then there exists an algebra isomorphism from $\mathcal{LA}(G_1)$ onto $\mathcal{LA}(G_2)$.*

Remark. The unique Φ and h found in the Corollary 3.5 indeed are T_1 and h resulted in Theorem 5.2.

6 bijective separating maps between Lebesgue-Fourier algebras on locally compact abelian groups

Let G be a locally compact abelian group. We know that the Fourier transform is an isometric algebra isomorphism between $A(\widehat{G})$ and $L^1(G)$ when \widehat{G} denotes the dual group of G ; moreover, the inverse Fourier transformation forms another isometric algebra isomorphism between $A(G)$ and $L^1(\widehat{G})$ (see [17], [13] and [14]). By the definition of $\mathcal{LA}(G)$ we can see that $\mathcal{LA}(G)$ with convolution (pointwise multiplication) is isometric isomorphic with $\mathcal{LA}(\widehat{G})$ with pointwise multiplication (convolution). These isometric isomorphisms lead to some results on $\mathcal{LA}(G)$ as a Banach algebra with convolution (see [10]). In this section G_1 and G_2 are assumed to be locally compact abelian groups.

Definition 6.1 *For linear map $T : \mathcal{LA}(G_1) \rightarrow \mathcal{LA}(G_2)$ if $f * g = 0$ implies that $Tf * Tg = 0$ for all $f, g \in \mathcal{LA}(G_1)$, we call T a separating map for convolution.*

Theorem 6.2 *Let $T : \mathcal{LA}(G_1) \rightarrow \mathcal{LA}(G_2)$ be a bijective separating map for convolution. Then T is continuous. Also \widehat{G}_1 and \widehat{G}_2 are homeomorphic.*

Proof. There exists a linear map \widehat{T} from $\mathcal{LA}(\widehat{G}_1)$ with pointwise multiplication onto $\mathcal{LA}(\widehat{G}_2)$ defined as $\widehat{T}\widehat{f} = \widehat{Tf}$, when \widehat{f} denotes the Fourier transformation of f for all $f \in \mathcal{LA}(G_1)$. It is clear that T is a separating map for convolution if and only if \widehat{T}

is a separating map.

Since T is a bijection, \hat{T} is also a bijective separating map. So we can use Theorem 3.3 for $\hat{T} : \mathcal{LA}(\hat{G}_1) \rightarrow \mathcal{LA}(\hat{G}_2)$. Therefore \hat{T} is continuous and hence T . Moreover, \hat{G}_1 and \hat{G}_2 are homeomorphic. \square

Since locally compact abelian groups are amenable, bijective separating maps for convolution have the decomposition that is mentioned in Theorem 5.2. So we can find $T_1 : \mathcal{LA}(G_1) \rightarrow \mathcal{LA}(G_2)$ an algebra isomorphism and $T_2 : \mathcal{LA}(G_2) \rightarrow \mathcal{LA}(G_2)$ a continuous linear transformation such that $T = T_2 \circ T_1$ where $T_2(g) = g * \mu$ for all $g \in \mathcal{LA}(G_2)$ when $\mu \in M(G_2)$.

According to the proof of Theorem 6.2 and Corollary 3.4, we can extend the bijective separating maps for convolution from Lebesgue-Fourier algebras to group algebras.

Corollary 6.3 *Let $T : \mathcal{LA}(G_1) \rightarrow \mathcal{LA}(G_2)$ be a bijective separating map for convolution. Then T has a unique continuous separating bijective extension $\dot{T} : L^1(G_1) \rightarrow L^1(G_2)$.*

The previous corollary and [10, Corollary 2] lead to an extension of bijective separating maps to the measure algebras of locally compact abelian groups.

Corollary 6.4 *Let $T : \mathcal{LA}(G_1) \rightarrow \mathcal{LA}(G_2)$ be a bijective separating map for convolution. Then T has a unique continuous separating bijective extension $\dot{T} : M(G_1) \rightarrow M(G_2)$.*

Conjecture. The existence of an extension of $T : \mathcal{LA}(G_1) \rightarrow \mathcal{LA}(G_2)$ - from Lebesgue-Fourier algebras - to Fourier-Stieltjes algebras for locally compact groups depends on finding a revised version of [17, Theorem 4.6.4] for general locally compact groups.

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References

- [1] Y. ABRAMOVICH, Multiplicative representation of disjointness preserving operators, *Indag. Math.* **45** (1983), 265-279.
- [2] J. ALAMINOS, M. BRESAR, J. EXTREMERA, AND A. R. VILLENA, Maps preserving zero products, *Studia Math.* **193** (2009), 131-159.
- [3] W. ARENDT, Spectral properties of Lamperti operators, *Indiana Univ. Math. J.* **32** (1983), 199-215.

- [4] E. BECKENSTEIN, L. NARICI, AND R. TODD, Automatic continuity of linear maps on spaces of continuous functions, *Manuscripta Math.* **62** (1988), 257-275.
- [5] DERIGHETTI, A. Some results on the Fourier-Stieltjes algebra of a locally compact group, *Comment. Math. Helv.* **45** (1970), 219-228.
- [6] W.R. EMERSON, AND F.P. GREENLEAF, Covering properties and Folner conditions for locally compact groups, *Math. Zeitschr.* **102**(1967), 370-384.
- [7] P. EYMARD, L'algebre de Fourier d'un groupe localement compacte, *Bull. Soc. Math. France* **92** (1964), 181-236.
- [8] J. J. FONT, Disjointness preserving mappings between Fourier algebras, *Colloq. Math.* **77** (1998), 179-187.
- [9] J. J. FONT AND S. HERNANDEZ, Algebraic characterization of locally compact groups, *J. Austral. Math. Sco.* **62** (1997), 405-420.
- [10] J. J. FONT AND S. HERNANDEZ, Automatic continuity and representation of certain linear isomorphisms between group algebras, *Indag. Math.* **6** (1995), 397-409.
- [11] J. J. FONT AND S. HERNANDEZ, Separating maps between locally compact spaces, *Arch. Math.* **63** (1994), 158-165.
- [12] F. GHAHRAMANI AND A. T. LAU, Weak amenability of certain classes of Banach algebras with out bounded approximate identities, *Math. Proc. Camb. Phil. Soc.* **133** (2002), 133-157.
- [13] E. HEWITT AND K. A. ROSS , Abstack harmonic analysis I, Springer-Verlag, Berlin, 1970.
- [14] E. HEWITT AND K. ROSS, Abstract harmonic analysis II, Springer-Verlag, Berlin, 1970.
- [15] M. S. MONFARED, Extensions and isomorphisms for the generalized Fourier algebras of a locally compact group, *J. Funct. Anal.* **198** (2003) 413-444.
- [16] J. R. MUNKRES, Topology, Prentice-Hall of India, New Delhi, 2007.
- [17] W. RUDIN, Fourier analysis on groups, Interscience Publishers, New York, 1962.

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